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PENNY-SHAPED CRACKS IN A FINITELY DEFORMED ELASTIC SOLID

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Abstract—The problem of a penny-shaped crack in an infinite, finitely deformed, compressible elastic material is considered. Solutions for a general form of strain–energy function are given for both circular and elliptic cracks when we have an equi-biaxial prestress in the plane of the crack. We also indicate how solutions to more general problems can be generated from previous work on elliptic cracks in linear anisotropic materials.

1. INTRODUCTION

Recently, Selvadurai[1] has considered the problem of a penny-shaped crack in an infinite, incompressible, finitely deformed elastic material. The crack is assumed to occupy the region $Z = 0$, $0 \leq R \leq A$ in the undeformed configuration and the finite equi-biaxial prestress is assumed to be homogeneous with cylindrical principal axes. In this note we consider the corresponding problem for a more general compressible material.

In Section 2 we derive the equilibrium equations and boundary conditions governing the problem. The incremental equations based on nominal stress and deformation gradient are used, see [2]. Following this we show that the potential methods developed for circular cracks in isotropic linear elastic materials, [3] for example, can be applied to this more general case. The solution for an arbitrary form of strain–energy function is obtained.

Shield[4] considered an elliptic crack in a linear, transversely isotropic elastic material and we find that the method used is applicable to the above finite deformation problem when the crack is elliptic rather than circular. We omit the details but the solution is given. While this does, of course, include the circular case the displacements are given in terms of an integral that is not particularly suitable for further analysis.

Finally we indicate how solutions to elliptic crack problems in a classical anisotropic elastic medium[5] can be used to obtain solutions to the problem of an elliptic crack in an isotropic hyperelastic material subjected to more general finite, homogeneous, deformations.

2. GOVERNING EQUATIONS

The body is assumed to undergo a finite homogeneous deformation such that the principal Cauchy stresses are

$$\sigma_r = \sigma_\theta, \quad \sigma_z = 0. \tag{1}$$

In terms of the strain–energy function $W(\lambda_1, \lambda_2, \lambda_3)$ we have

$$J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} \quad (i = 1, 2, 3) \tag{2}$$

where subscripts (1, 2, 3) correspond to the (r, θ, z) directions respectively, $\lambda_i (i = 1, 2, 3)$ are the principal stretches ($\lambda_1 = \lambda_2 = \lambda$ say) and $J = \lambda_1\lambda_2\lambda_3$. From (1) and (2) we have an implicit equation for $\lambda_3(\lambda)$, namely

$$\frac{\partial W}{\partial \lambda_3} \equiv W_3(\lambda, \lambda_3) = 0.$$

The incremental equations can be written

$$\operatorname{div} \dot{\mathbf{s}} = \mathbf{0}, \tag{3}$$

where div is the divergence operator in the current, finitely deformed configuration and $\dot{s}$ is the increment of the (asymmetric) nominal stress tensor evaluated in the current configuration, see [2] for details. We have

$$\dot{\mathbf{s}} = \mathbf{B}\boldsymbol{\eta}, \tag{4}$$

where the non-zero components of the fourth order tensor of instantaneous moduli are

$$\left.\begin{aligned} JB_{iijj} &= \lambda_i\lambda_j \frac{\partial^2 W}{\partial\lambda_i\partial\lambda_j}, \\ B_{ijij} &= \frac{(\sigma_i - \sigma_j)\lambda_i^2}{\lambda_i^2 - \lambda_j^2} & i \neq j,\ \lambda_i \neq \lambda_j, \\ B_{ijij} &= \frac{1}{2}(B_{iiii} - B_{iijj} + \sigma_i) & i \neq j,\ \lambda_i = \lambda_j, \\ B_{ijji} &= B_{jiij} = B_{ijij} - \sigma_i & i \neq j, \end{aligned}\right\} \tag{5}$$

and $\boldsymbol{\eta}$ is the incremental displacement gradient.

If we write $u(r, z)$, $w(r, z)$ for the incremental displacements in the radial and axial directions respectively the incremental equations (3) for an axisymmetric displacement field in cylindrical coordinates can be written

$$B_{1111}(u_{rr} + u_r/r - u/r^2) + B_{3131}u_{zz} + (B_{1133} + B_{3131})w_{rz} = 0, \tag{6}$$

$$(B_{1133} + B_{3131})(u_{rz} + u_z/r) + B_{1313}(w_{rr} + w_r/r) + B_{3333}w_{zz} = 0, \tag{7}$$

having used (3) with (4), (5) and (1). The subscripts r, z denote partial differentiation.

To obtain solutions to (6) and (7) we introduce the potential function $\phi(r, z)$ defined by

$$\left.\begin{aligned} u &= -\{(B_{1133} + B_{3131})/B_{3131}\}\phi_{rz}, \\ w &= B_{1111}(\phi_{rr} + \phi_r/r)/B_{3131} + \phi_{zz}. \end{aligned}\right\} \tag{8}$$

In this case (6) is satisfied identically. Substituting (8) into (7) leads to the equation

$$\nabla_1^2\nabla_2^2\phi = 0, \tag{9}$$

where the operators ∇_i^2 are defined by

$$\nabla_i^2 \equiv \alpha_i^2\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\ (i = 1, 2) \tag{10}$$

and α_1^2, α_2^2 are the roots of the quadratic

$$B_{3333}B_{3131}\alpha^4 + \{(B_{1133} + B_{3131})^2 - B_{1111}B_{3333} - B_{1313}B_{3131}\}\alpha^2 + B_{1111}B_{1313} = 0. \tag{11}$$

We note that the potential function defined by (8) is the *only* potential function of that form which will allow *either* (6) or (7) to be satisfied identically. Since ϕ will not permit the incompressibility condition to be satisfied it is clear that the problem for an incompressible material will require a different method of solution, see [1].

We suppose that the crack is opened by a constant pressure P so that the required boundary conditions are

$$\left.\begin{array}{ll} \dot{s}_{31}(r,0)=0, & 0\leqslant r<\infty, \\ \dot{s}_{33}(r,0)=-P, & 0\leqslant r\leqslant a, \\ w(r,0)=0, & a\leqslant r<\infty. \end{array}\right\} \tag{12}$$

Henceforth we consider only the half-space $z\geqslant 0$, solutions for the lower half-plane may be obtained by symmetry.

3. SOLUTION

By taking zero-order Hankel transforms of (9), with respect to r, we have

$$\bar{\phi}(\xi,z)=c_1\,e^{\alpha_1\xi z}+c_2\,e^{-\alpha_1\xi z}+c_3\,e^{\alpha_2\xi z}+c_4\,e^{-\alpha_2\xi z}\quad(\alpha_1\neq\alpha_2) \tag{13}$$

where

$$\bar{\phi}(\xi,z)=\int_0^\infty r\phi(r,z)J_0(\xi r)\,dr, \tag{14}$$

J_0 being the Bessel function of the first kind of order zero, and $c_i(i=1,2,3,4)$ are arbitrary functions of ξ. We suppose that α_1, α_2 have positive real parts so that $c_1=c_3=0$ to ensure finite solutions as $z\to\infty$. By taking the inverse transform of (13) with $(12)_1$, (4) and (8) we find that

$$c_4(\xi)=-c_2(\xi)\{B_{1111}+B_{1133}\alpha_1^{\,2}\}/\{B_{1111}+B_{1133}\alpha_2^{\,2}\}. \tag{15}$$

Similarly $(12)_3$ leads to the condition

$$\int_0^\infty c_2(\xi)\xi^3J_0(\xi r)\,d\xi=0\quad a\leqslant r<\infty, \tag{16}$$

and $(12)_2$ gives

$$\int_0^\infty c_2(\xi)\xi^4J_0(\xi r)\,d\xi=-PF\quad 0\leqslant r\leqslant a, \tag{17}$$

where

$$F=\{B_{1111}+B_{1133}\alpha_2^{\,2}\}/\{(\alpha_1-\alpha_2)[G(B_{1111}-\alpha_1\alpha_2B_{1133}) - B_{3333}(\{\alpha_1^{\,2}+\alpha_1\alpha_2+\alpha_2^{\,2}\}B_{1111}+\alpha_1^{\,2}\alpha_2^{\,2}B_{1133})]\}, \tag{18}$$

and

$$G=\{B_{1111}B_{3333}-B_{1133}(B_{1133}+B_{3131})\}/B_{3131}. \tag{19}$$

General dual integral equations of the type (16) and (17) have been solved by Sneddon[3, 6] for example, and we find that

$$c_2(\xi)=-2PF\,(\sin a\xi-a\xi\cos a\xi)/\pi\xi^5. \tag{20}$$

Substituting (20) into (13) and (14) gives, upon taking the inverse transform

$$w(r,z)=2PF^*\int_0^\infty \xi^{-2}J_0(\xi r)[a\xi\cos a\xi-\sin a\xi]\{[B_{1111}+B_{1133}\alpha_2^{\,2})\times(B_{3131}\alpha_1^{\,2}-B_{1111})\,e^{-\alpha_1\xi z}-(B_{1111}+B_{1133}\alpha_1^{\,2})(B_{3131}\alpha_2^{\,2}-B_{1111})\,e^{-\alpha_2\xi z}\}\,d\xi, \tag{21}$$

$$u(r,z) = -(B_{1133}+B_{3131})2PF^*\int_0^\infty \xi^{-2}J_1(\xi r)[a\xi\cos a\xi - \sin a\xi]$$
$$\times\{\alpha_2(B_{1111}+B_{1133}\alpha_1^2)\,e^{-\alpha_2\xi z} - \alpha_1(B_{1111}+B_{1133}\alpha_2^2)\,e^{-\alpha_1\xi z}\}\,d\xi, \tag{22}$$

where

$$F^* = F/\pi B_{3131}(B_{1111}+B_{1133}\alpha_2^2).$$

In particular, for $z = 0$, we have

$$\left.\begin{aligned} w(r,0) &= 2PF^*(a^2-r^2)^{1/2}(\alpha_2^2-\alpha_1^2)B_{1111}(B_{1133}+B_{3131}),\ r\le a\\ u(r,0) &= 2\pi PF^* r(\alpha_1-\alpha_2)(B_{1111}-\alpha_1\alpha_2B_{1133})(B_{1133}+B_{3131}).\end{aligned}\right\} \tag{23}$$

To make comparisons with previous work we now apply the Griffith criterion for crack growth. It is easily shown that the crack will extend when the pressure reaches a critical value

$$P_c = \{S/aF^*(\alpha_2^2-\alpha_1^2)B_{1111}(B_{1133}+B_{3131})\}^{1/2}, \tag{24}$$

where S is the surface tension of the material. The value of P_c for an incompressible material was given in [1]. If we consider the special case $\lambda = 1$ then a limiting proceedure gives

$$P_c = \{\pi\mu S/A(1-\nu)\}^{1/2}$$

where μ, ν are the ground state shear modulus and Poisson's ratio respectively. This is the result given in [3] for a linear isotropic elastic material.

If we set $P_c \equiv 0$ then (24) provides an equation for the critical value of λ at which the surface of a radially compressed half-space may warp out of the plane. This *bifurcation* problem has been considered in the more general context ($\lambda_1 \neq \lambda_2$) in [7].

4. AN ELLIPTIC CRACK

We now consider the problem of Section 2 when the crack is elliptic rather then circular. If we write the general incremental equations, (3) with (4), for a *homogeneous* finite deformation in Cartesian coordinates, (x_1, x_2, x_3) in the current configuration, we have

$$B_{ijkl}u_{l,ik} = 0,\ (j = 1,2,3) \tag{25}$$

where the usual notation and conventions are adopted and $\mathbf{u} = (u_1, u_2, u_3)$ is the incremental displacement vector. It is obvious that there are similarities between (25) and the equilibrium equations for a linear anisotropic elastic material, but the equations are not the same since $B_{ijkl} \neq B_{jikl}$, $B_{ijkl} \neq B_{ijlk}$ in general. It is well known, however, that the methods developed for anistropic elastic materials may often be successfully applied to the "corresponding" incremental problem. For example, the incremental equations (6) and (7) have five independent instantaneous moduli. Similarly, a transversely isotropic material has five elastic moduli and so we may expect some similarities between the above finite deformation problem for an elliptic crack and the classical transversely isotropic problem [4]. We find that the method of [4] can in fact be applied to this problem. However, symmetries of the classical elastic moduli not available to the instantaneous moduli were employed and so it is not merely a matter of comparing coefficients. We omit details but give the solution.

If, in Cartesian coordinates we assume the crack to occupy the region

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}\le 1,\ x_3 = 0 \tag{26}$$

in the current configuration and define

$$\Phi(x_1,x_2,x_3) = \frac{a_1a_2^2P}{4\beta E(K)}\int_\gamma^\infty\left\{\frac{x_1^2}{a_1^2+s}+\frac{x_2^2}{a_2^2+s}+\frac{x_3^2}{s}-1\right\}\{s(a_1^2+s)(a_2^2+s)\}^{-1/2}\,ds,$$

where

$$\beta = B_{3333}\left\{\frac{k_1}{\nu_1(k_1+1)} - \frac{k_2}{\nu_2(k_2+1)}\right\} + B_{1133}\left\{\frac{\nu_2}{k_2+1} - \frac{\nu_1}{k_1+1}\right\},$$

k_1, k_2 being the roots of

$$\frac{B_{3131} + k(B_{1133} + B_{3131})}{B_{1111}} = \frac{kB_{3333}}{kB_{1313} + B_{1133} + B_{3131}} = \nu^2,$$

$$E(K) = \int_0^K \mathrm{d}n^2 t \,\mathrm{d}t$$

where K is the $\frac{1}{4}$ period of the Jacobian elliptic function *sn* corresponding to the modulus

$$m = (a_1{}^2 - a_2{}^2)^{1/2}/a_1$$

and, finally, γ is the positive elliptic coordinate, i.e. the positive root of

$$\frac{x_1{}^2}{a_1{}^2+s} + \frac{x_2{}^2}{a_2{}^2+s} + \frac{x_3{}^2}{s} - 1 = 0,$$

and if we are inside the ellipse $\gamma = 0$. Then the displacements are given by

$$u_1 = \phi_{,1} + \psi_{,1},\ u_2 = \phi_{,2} + \psi_{,2},\ u_3 = k_1\phi_{,3} + k_2\psi_{,3}$$

where

$$\phi(x_1, x_2, x_3) = \frac{\nu_1}{k_1+1}\Phi(x_1, x_2, x_3/\nu_1),$$

and

$$\psi(x_1, x_2, x_3) = \frac{-\nu_2}{k_2+1}\Phi(x_1, x_2, x_3/\nu_2).$$

While this solution contains the special case of a circular crack the direct approach of Section 3 gives the displacements in a more suitable form for further analysis.

5. FURTHER PROBLEMS

If we wish to consider problems with less symmetry than that above, in particular the case $\lambda_1 \neq \lambda_2$, then potential methods are no longer available and an alternative method of solution is required. Willis [5] has given solutions to the problem of an elliptic crack in a general anistropic linear elastic medium subjected to polynomial loading at infinity, the resultant displacement are expressed in terms of a contour integral. Provided we have

$$\det [B_{ijkl} n_i n_k] \neq 0 \tag{27}$$

for all unit vectors $\mathbf{n}$, and we note that (27) is implied by the strong ellipticity condition

$$B_{ijkl} a_i b_j a_k b_l > 0 \ \forall \text{ non-zero vectors } \mathbf{a}, \mathbf{b},$$

then we need only replace c_{ijkl} in [5] by B_{jilk} to obtain solutions to the corresponding incremental problem.

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